Supplementary Information: Super-resolution imaging of azimuthal features with illumination carrying OAM

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This document contains supplementary information in support of the manuscript "Super-resolution imaging of azimuthal features with illumination carrying OAM".

I. MATHEMATICAL DERIVATION

The detail mathematical derivation in support of the main paper is presented here. This supplemental material is organised in two parts, the mathematical derivation in Sec. I and quantification in Sec. II. Inside the Sec. I, at first we discuss about the basics of imaging I A, following the mathematics behind the resolution enhancement in spatial I B and angle-OAM degree of freedom I C, then connect it with a nice physical intuitive model in the discussion I E.

A. Basics of Imaging

We start our discussion with the basic mathematical formulation behind a simple imaging system. We consider (x'', y'') is the object plane, (x', y') is the lens plane and (x, y) is the image plane as shown in the Fig. 1. We begin by defining the electric field at the object plane as $E(x'', y''; z = 0) = E_{in}(x'', y'', z = 0)T(x'', y'')$, where, $E_{in}(x'', y'', z = 0)$ represents illuminating field and $T(x'', y'')$ is the object transmission function. Next, we employ Fresnel propagation to describe the evolution of the field from the object plane to a distance of $z = 2f$ at (x', y') plane. It is given by,

$$E(x', y'; z = 2f) = e^{-\frac{ik}{4f}(x'^2+y'^2)} \iint E(x'', y''; z = 0)e^{-\frac{ik}{4f}(x''^2+y''^2)}e^{\frac{ik}{2f}(x'x''+y'y'')}dx''dy''. \quad \text{(S1)}$$

Consider the lens transmission function $T_L = e^{-\frac{(x'^2+y'^2)}{2d^2}} e^{\frac{ik}{2f}(x'^2+y'^2)}$ where d is the effective aperture size, and f is the focal length of the lens respectively. The electric field amplitude just after the lens is given by

$$E_{lens}(x', y'; z = 2f) = e^{-\frac{(x'^2+y'^2)}{2d^2}} e^{\frac{ik}{2f}(x'^2+y'^2)}E(x', y'; z = 2f). \quad \text{(S2)}$$

We then apply Fresnel propagation from the lens plane to the image plane at $z = 4f$ and we can write it as,

$$E(x, y; z = 4f) = e^{-\frac{ik}{4f}(x^2+y^2)} \iint E_{lens}(x', y'; z = 2f)e^{-\frac{ik}{4f}(x'^2+y'^2)}e^{\frac{ik}{2f}(xx'+yy')}dx'dy', \quad \text{(S3)}$$

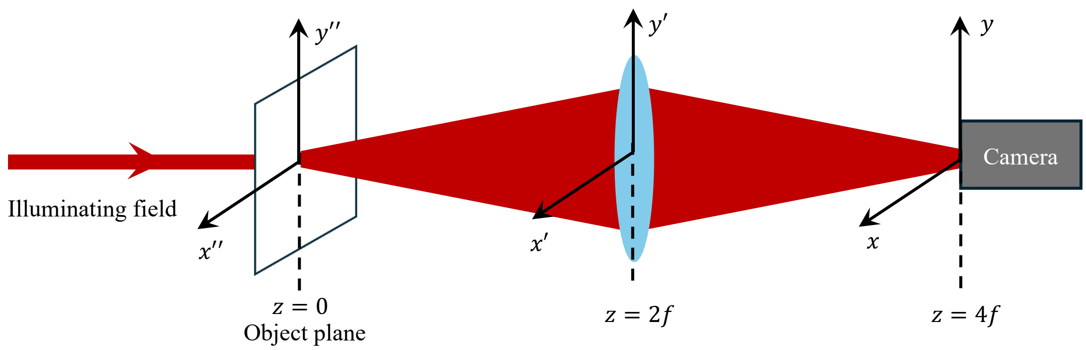

FIG. 1. Schematic of a setup to image an object using a $2f - 2f$ imaging system.

$$E(x,y;z=4f) = e^{-\frac{ik}{4f}(x^2+y^2)} \iint E_{in}(x'',y'',z=0)T(x'',y'')e^{-\frac{ik}{4f}(x''^2+y''^2)} \times \left[\int e^{-\frac{x'^2}{2d^2}+\frac{ik}{2f}(x+x'')x'}dx'\right]\left[\int e^{-\frac{y'^2}{2d^2}+\frac{ik}{2f}(y+y'')y'}dy'\right]dx''dy''. \quad \text{(S4)}$$

Now we use the standard gaussian integration formula $\int e^{-\alpha x^2+\beta x}dx = \left(\frac{\pi}{\alpha}\right)^{1/2} e^{\left(\frac{\beta^2}{4\alpha}\right)}$ to solve the two integral inside, which becomes,

$$\int e^{-\frac{x'^2}{2d^2}+\frac{ik}{2f}(x+x'')x'}dx' \approx e^{-\frac{d^2k^2}{4f^2}(x+x'')^2} \text{and} \qquad \int e^{-\frac{y'^2}{2d^2}+\frac{ik}{2f}(y+y'')y'}dy' \approx e^{-\frac{d^2k^2}{4f^2}(y+y'')^2}. \quad \text{(S5)}$$

After substituting and performing the required standard integrations, we simplify the equation to obtain the electric field at the image plane as,

$$E(x,y;z=4f) = e^{-\frac{ik}{4f}(x^2+y^2)} \iint E_{in}(x'',y'',z=0)T(x'',y'')e^{\frac{ik}{4f}(x''^2+y''^2)}e^{-\frac{k^2d^2}{4f^2}(x+x'')^2} \times e^{-\frac{k^2d^2}{4f^2}(y+y'')^2}dx''dy''. \quad \text{(S6)}$$

We write the intensity distribution as $I_l(x,y,z=4f) = |E(x,y,z=4f)|^2$, and substituting $\frac{2f}{kd}$ as σ_p, which leads to

$$I_l(x,y,z=4f) = e^{-\frac{x^2+y^2}{\sigma_p^2}} \left| \iint E_{in}(x'',y'',z=0)T(x'',y'') \exp\left[-\frac{(x''^2+y''^2)}{2\sigma_p^2}\left(1 - i\frac{k\sigma_p^2}{2f}\right)\right] \times \exp\left[-\frac{xx''+yy''}{\sigma_p^2}\right]dx''dy''\right|^2. \quad \text{(S7)}$$

B. Resolution in Spatial degree of freedom

The input field is a gaussian plane wave, which can be written as $E_{in}(x'', y'') = e^{i(k_x'' x'' + k_y'' y'')} e^{-\frac{(x''^2 + y''^2)}{2w^2}}$. For the simplest case, considering the object as two dirac delta like slits separated by a distance s in x-direction only i.e., $T(x'', y'') = T(x'') = \delta(x'' - \frac{s}{2}) + \delta(x'' + \frac{s}{2})$. By substituting $E_{in}(x'', y'')$ and $T(x'', y'')$ in Eq. S7 and writing $\left(1 + \frac{\sigma_p^2}{2w^2} - i\frac{k\sigma_p^2}{2f}\right)$ as a, then we get,

$$I(x, y)_{z=4f} = \left| f\left(x - \frac{s}{2}, y\right) e^{ik_x'' s/2} + f\left(x + \frac{s}{2}, y\right) e^{-ik_x'' s/2} \right|^2 \tag{S8}$$

$$\text{where,} \quad f\left(x - \frac{s}{2}, y\right) = e^{-\frac{s^2 a}{8\sigma_p^2}} e^{-xs/2} \int e^{-\frac{y''^2 a}{2\sigma_p^2}} e^{-\frac{yy''}{\sigma_p^2}} e^{ik_y y''} dy'' \tag{S9a}$$

$$\text{and} \quad f\left(x + \frac{s}{2}, y\right) = e^{-\frac{s^2 a}{8\sigma_p^2}} e^{xs/2} \int e^{-\frac{y''^2 a}{2\sigma_p^2}} e^{-\frac{yy''}{\sigma_p^2}} e^{-ik_y y''} dy'' \tag{S9b}$$

Now at $x = 0$, that is at the mid-point between the two slits, the intensity $I(x, y)_{z=4f}$ becomes

$$I(x = 0, y)_{z=4f} = |f(-\frac{s}{2}, y)|^2 + |f(\frac{s}{2}, y)|^2 + 2f(-\frac{s}{2}, y) f(\frac{s}{2}, y) \cos(k_x'' s). \tag{S10}$$

Here we have assumed the function to be real, thus we can write $f(-\frac{s}{2}, y) = f(\frac{s}{2}, y) = f_0$. Hence we obtain

$$I(x = 0, y)_{z=4f} = 2|f_0|^2 (1 + \cos(k_x'' s)). \tag{S11}$$

When $cos(k_x'' s) = -1$, then the intensity exactly vanishes at the midpoint between the slits. This situtation corresponds to the point of maximum resolution. Extending this argument further we can express it as follows:

$$cos(k_x'' s) = -1 \implies k_x'' s = (2n + 1)\pi \implies sk_0 \sin\theta = (2n + 1)\pi, \tag{S12}$$

where, θ is the angle of the incident wave w.r.t. the normal (see Fig. 2). This formula has a practical limitation. As the angle θ increases, the tilt of the incident wavefront becomes more pronounced. However, beyond a certain threshold of θ, the corresponding shift in Fourier space grows excessively, leading to image distortion. If we consider an object with both $x-y$ features and calculate the above formula, then there will be another term corresponding to the k_y''. And if tilting in one direction enhances the resolution in particular direction (say x) then it will be compensated with the resolution degradation in another direction. So,

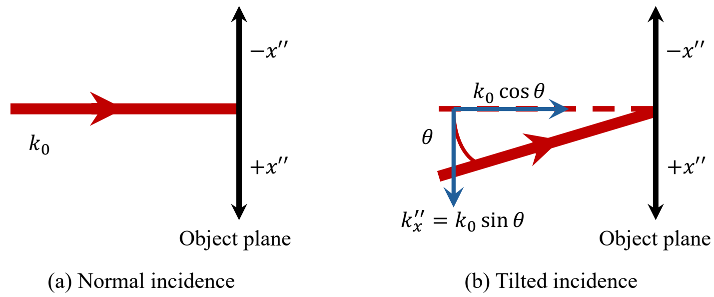

FIG. 2. Schematic representation of the tilting geometry.

tilting does not magically increase resolution. If we collect multiple tilted illuminations and combine them, then overall resolution enhancement emerges. This fundamental concept is actually being used for structured illumination microscopy(SIM)[1,2], Fourier ptychography (FT)[3], synthetic aperture/off-axis illumination[4]. With Eq. S12 we represent a simple way of understanding how tilting a plane wave can help in resolution enhancement in a particular direction. For a practical three-dimensional case it is very intuitive in the $k-$space. If we consider a sphere with three axes k_x, k_y, k_z then $|k|$ is the radius and for a particular wavelegth beam this $|k|$ is constant. Then basically tilting corresponds to moving along the surface of this sphere.

C. Resolution in azimuthal degree of freedom

Here we detail the mathematical derivation to describe the reasons of getting better visibility using OAM and cause of 100% visibility using the similar toy model as it is used for the spatial case above (this part is already discussed in the main paper). We consider that the object only has the azimuthal features, that is $T(r'', \theta'') = T(\theta'')$. For this, we consider an azimuthal-double-slit object of infinitesimally narrow slit width and angular separation of $2\theta_0$ such that the transmission function is given by $T(\theta'') = [\delta(\theta'' - \theta_0) + \delta(\theta'' + \theta_0)]$. Now if we write S7 in polar coordinate, then it becomes

$$I(r, \theta, z = 4f) = Ae^{\left(-\frac{r^2}{\sigma_p^2}\right)} \left| \iint T(\theta'') e^{il\theta''} \exp\left[-\frac{ar''^2}{2\sigma_p^2}\right] \times \exp\left[-\frac{r''r}{\sigma_p^2} \cos(\theta - \theta'')\right] r''dr''d\theta'' \right|^2,$$

where again $\left(1+\frac{\sigma_p^2}{2w^2}-i\frac{k\sigma_p^2}{2f}\right)$ is denoted as a. The image-plane intensity $I(r,\theta,z=4f)$ can be expressed as

$$I(r,\theta,z=4f) = Ae^{-\frac{r^2}{\sigma_p^2}}\left|\iint\left[\delta(\theta''-\theta_0)+\delta(\theta''+\theta_0)\right]e^{il\theta''}\right.$$
$$\left.\times\exp\left[-\frac{ar''^2}{2\sigma_p^2}\right]\exp\left[-\frac{r''r}{\sigma_p^2}\cos(\theta-\theta'')\right]r''dr''d\theta''\right|^2.$$

After applying the dirac delta function operation we obtain

$$I(r,\theta,z=4f) = \left|f(r,\theta-\theta_0)e^{il\theta_0}+f(r,\theta+\theta_0)e^{-il\theta_0}\right|^2; \tag{S13}$$

$$\text{where}\quad f(r,\theta) = Ae^{-\frac{r^2}{\sigma_p^2}}\int\exp\left[-\frac{ar''^2}{2\sigma_p^2}\right]\exp\left[-\frac{r''r}{\sigma_p^2}\cos\theta\right]r''dr''. \tag{S14}$$

Now, at $\theta=0$, that is, at the mid-point between the two slits, S13 can be written as:

$$\begin{aligned} I(r,\theta=0,z=4f) &= |f(r,-\theta_0)|^2+|f(r,\theta_0)|^2+f(r,-\theta_0)f(r,\theta_0)(e^{i2l\theta_0}+e^{-i2l\theta_0}). \\ &= |f(r,-\theta_0)|^2+|f(r,\theta_0)|^2+2f(r,-\theta_0)f(r,\theta_0)\cos 2l\theta_0. \end{aligned} \tag{S15}$$

Here, we have assumed the function $f(r,\theta)$ to be real, thus we write $f(r,-\theta_0)=f(r,\theta_0)=f_0$. Thus the above equation is written as

$$I(r,\theta=0,z=4f) = 2|f_0|^2(1+\cos 2l\theta_0). \tag{S16}$$

For maximum image contrast, the intensity at the midpoint must vanish, i.e.,$I(r,\theta=0,z=4f)=0$. This condition is satisfied when $\cos(2l\theta_0)=-1$, which leads to the condition deduced below:

$$cos(2l\theta_0)=-1 \implies 2l\theta_0=(2n+1)\pi \implies 2l\theta_0=(2n+1)\pi.$$

Finally it determines the optimum OAM mode index l_{opt} that yields the highest contrast:

$$l_{\text{opt}} = \frac{(2n+1)\pi}{2\theta_0}. \tag{S17}$$

Here, $2\theta_0$ is the angular separation between the two angular slits. If we consider $2\theta_0=\alpha$, then it can be rewritten as $l\alpha=(2n+1)\pi$.

D. Angular Resolution Criterion

According to the Rayleigh resolution criterion[5], the minimum resolvable separation between two point sources under incoherent illumination is defined by the condition $I_{\rm min} = 0.81\, I_{\rm max}$. In the present case, the maximum intensity $I_{\rm max}$ occurs at the locations of the two slits, i.e., at $\theta = \pm\theta_0$, where the individual intensity contributions are highest. Since the illumination is fully incoherent, the interference term (the third term in Eq. S15) does not contribute. Therefore, the maximum intensity can be written as

$$I_{\rm max} = |f(r, \theta - \theta_0)|^2 + |f(r, \theta + \theta_0)|^2,$$

where $f(r, \theta)$ is defined in Eq. S14. The minimum intensity $I_{\rm min}$ occurs at the midpoint

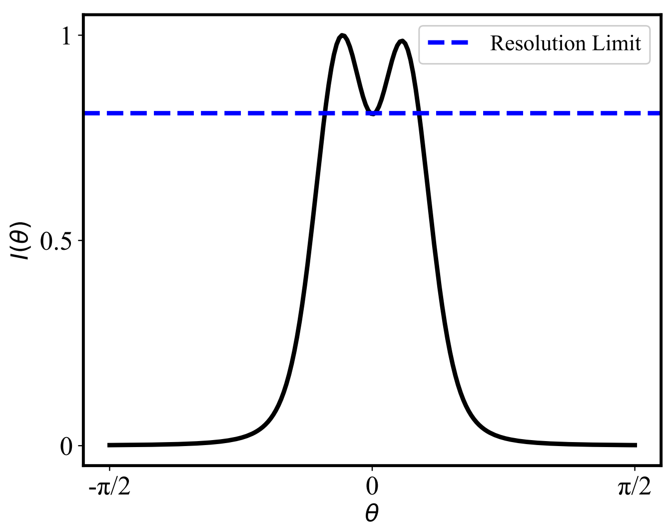

FIG. 3. The Rayleigh resolution limit corresponding to the experimental setup indicates a minimum resolvable angular separation of $\alpha = \pi/8$ between two azimuthal slits (or points).

between the two slits, corresponding to $\theta_0 = 0$. At this location, the intensity becomes

$$I_{\rm min} = 2|f(r, \theta)|^2.$$

Substituting these expressions into the Rayleigh criterion yields

$$\frac{2|f(r, \theta)|^2}{|f(r, \theta - \theta_0)|^2 + |f(r, \theta + \theta_0)|^2} = 0.81. \tag{S18}$$

Solving this equation provides the minimum resolvable angular separation ($2\theta_0$), which depends on the imaging parameters such as wavelength, focal length, and aperture, encapsulated in σ_p within the function f, where $\sigma_p = \frac{2f}{kd}$. Since the resulting expression is transcendental, an analytical solution is not feasible; therefore, we solve it numerically for the given set of experimental parameters which are used in the experiment.

For our setup, the Rayleigh criterion yields a minimum azimuthal separation of $\frac{\pi}{8}$ with a visibility of approximately 10.49% [see Fig. 3]. In contrast, our method achieves a separation of $\frac{\pi}{10}$ with 100% visibility. This demonstrates sub-Rayleigh resolution with nearly an order-of-magnitude enhancement (10 times) in visibility, enabling high-contrast imaging beyond the classical diffraction limit.

E. Discussion

By comparing Eq. S12, corresponding to spatial imaging, with Eq. S17 for angular imaging, a direct analogy can be established. In the spatial domain, increasing the tilt of the illuminating wavefront enhances resolution for spatially structured objects. In an analogous manner, in the angular domain, higher OAM mode indices (l) lead to improved resolution when imaging azimuthally structured objects. This naturally introduces the concept of selecting optimal structured beams tailored to the symmetry of the object. For a transverse double-slit, it is well known that increasing the relative phase difference between the two slits from 0 to π enhances the image contrast, reaching its maximum at π, where the intensity at the midpoint vanishes[6].

Similarly, in the case of two azimuthal slits, an OAM beam—characterized by an azimuthally varying phase—can be tuned to induce a π phase difference between the slits, thereby achieving maximum contrast. In this framework, the relative phase difference is given by $\Delta\phi = 2l\theta_0$, which can be controlled through the OAM mode index l. Thus, by varying l, the phase difference can be tuned from 0 to π. For example, for a slit separation of $2\theta_0 = 0.1\pi$, choosing $l = 10$ yields a π phase difference between adjacent azimuthal slits, resulting in 100% contrast. This physical interpretation is further illustrated schematically in Fig. 3. Within this perspective, the Rayleigh resolution condition can be interpreted as the case where the relative phase difference is $\pi/2$, corresponding to the point at which the interference term vanishes, consistent with the classical resolution criterion. This interpretation is comparable to the second case in the Fig. 4.

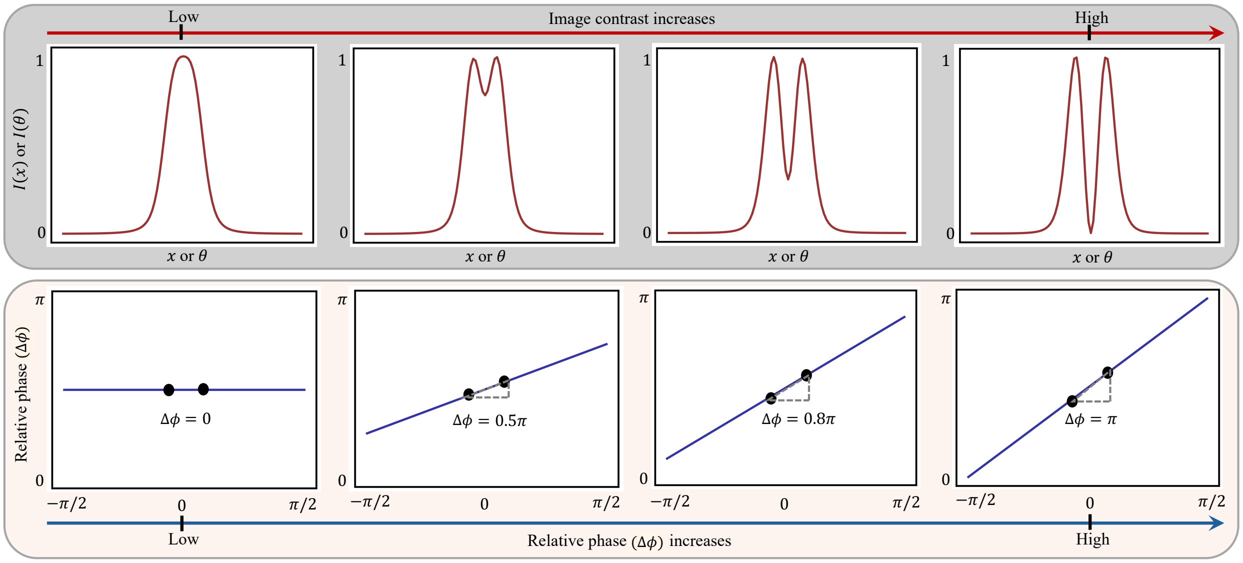

FIG. 4. Visualization of physical understanding which shows that with the relative phase image contrast increases whereas it reaches to the highest resolution when relative phase is π. Here $\Delta\phi = 2l\theta_0$ is the relative phase difference between the light incident on adjacent slits.

We emphasize on a fact that, in the spatial case, controlling the tilt of the coherent illumination to reveal such periodicity is not practical, since beyond a certain angle the Fourier frequency shift becomes abrupt, ultimately degrading resolution and distorting the image. In contrast, because l directly governs phase structuring, OAM beams naturally exhibit this periodic resolution enhancement in a well-controlled manner.

II. QANTIFICATION: FISHER INFORMATION AND CRAMÉR–RAO LOWER BOUND

For objects exhibiting complex angular features, visibility by itself is insufficient as a measure of image quality. A more rigorous assessment can be obtained through the evaluation of Fisher Information (FI) and the associated Cramér–Rao Lower Bound (CRLB)[7,8]. Since the intensity distribution at the image plane acts as a probability density function (PDF), i.e., $p(\theta \mid \theta_0) = I(\theta, z = 4f)$, where θ_0 represents the angular separation to be estimated, we define FI ($F(\theta_0)$) and CRLB (Δ) as follows:

$$F(\theta_0) = \int \left[\frac{\partial}{\partial\theta_0}\log p(\theta \mid \theta_0)\right]^2 p(\theta \mid \theta_0)d\theta, \tag{S19}$$

$$\Delta \geq \frac{1}{F(\theta_0)}. \tag{S20}$$

For objects like the Siemens star, where determining $I_{\max}$ and $I_{\min}$ is ambiguous, due to both experimental noise and complexity of the object. Thus visibility is not a suitable metric. Therefor, we also compute FI and CRLB, which provide a more meaningful quantification. We observe that the minimum CRLB corresponding the lowest variance in estimating θ_0 occurs for $l = 10$, which aligns with our theoretical model. Here we roughly compute FI by doing single parameter estimation but for more precise quantification two-parameter or multi-parameter estimation would be more suitable.

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